Topological Defn of Compact.

A set $C \subseteq \mathbb{R}$ is compact $\iff$ Every open cover of C has a finite subcover.

Given a set $S \subseteq \mathbb{R}$, an open cover of S is a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of open sets in $\mathbb{R}$

index set

s.t. $S \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$

Given an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of S, we say U_α has a finite subcover if $\exists \{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ s.t. $S \subseteq \bigcup_{j=1}^{k} U_{\alpha_j}$.

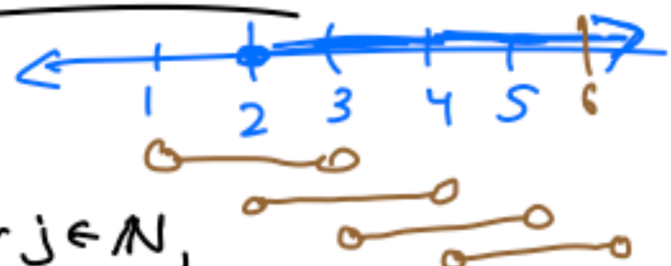
<u>Example:</u> The set $\{0,1\}$ is (topologically) compact.

<u>Pf:</u> Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be a cover of $\{0,1\}$. Then $\exists\, \beta\in\mathcal{A}$ s.t. $0\in U_\beta$, and $\exists\, \gamma\in\mathcal{A}$ s.t. $1\in U_\gamma$. Then $\{U_\beta, U_\gamma\}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$.

Thus $\{0,1\}$ is (topologically) compact. $\blacksquare$

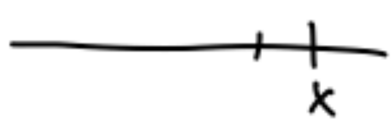
<u>Example</u>: The set $[2,\infty)$ is <u>not</u> topologically compact.

<u>Pf.</u> Let



$U_j = (j, j+2)$ for $j\in\mathbb{N}$,

and observe that $\{U_j\}_{j\in\mathbb{N}}$ is an open cover of $[2,\infty)$, because if $x\in[2,\infty)$, $x\geq 2$, and



$U_{\lfloor x\rfloor -1} = (\lfloor x\rfloor -1, \lfloor x\rfloor +1)$ contains x. $\left(\begin{array}{c} x\geq 2 \\ \lfloor x\rfloor \geq 2 \\ \lfloor x\rfloor -1\geq 1 . \checkmark \end{array}\right)$

$$\left(\lfloor x \rfloor - 1 < x - 1 < x < \lfloor x \rfloor + 1 \right)$$
$$\forall x \in \mathbb{R}.$$

This open cover has no finite subcover of $[2, \infty)$. To see this, suppose that $\{U_{j_1}, \ldots, U_{j_k}\}$ is a finite subset of $\{U_j\}_{j \geq 1}$. Then $\forall x \in \bigcup_{p=1}^{k} U_{j_p}$,

$$x < \max\{j_1, \ldots, j_k\} + 2.$$

Thus the subcover can't cover all of $[2, \infty)$. $\boxed{/\!/}$

$\therefore [2, \infty)$ is not top. compact

Big Theorem (Heine-Borel Theorem)

A set $C \subseteq \mathbb{R}$ is (sequentially) compact $\iff$ C is (topologically) compact.

That is, the following 3 criteria are equivalent.

1. Every sequence of points in C has a subsequence converging to a point of C.
2. C is closed & bound.
3. Every open cover of C has a finite subcover.

Proof is in book.

Example Lemma.

A closed subset of a compact set is compact.

Pf. Let $C \subseteq \mathbb{R}$, $F \subseteq \mathbb{R}$ where C is compact and F is closed. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be any open cover of $F \cap C$. Then $\{U_\alpha\}_{\alpha \in \mathcal{A}} \cup \{F^c\}$ is an open cover of C, since

$$C = (C \cap F) \cup (C \cap F^c) \text{ and}$$

so $C \cap F^c \subseteq F^c$,

$$C \cap F \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha .$$

Thus, since C is compact, $\exists$ a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_k}, F^c\}$

$\subseteq \{U_{\alpha_1}, \dots, U_{\alpha_k}, F^c\}$. possibly

Thus $C \subseteq \bigcup_{j=1}^{k} U_{\alpha_j} \cup F^c$

Then $C \cap F \subseteq \bigcup_{j=1}^{k} U_{\alpha_j}$. Thus, we have a finite subcover of $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of $C \cap F$. Therefore $C \cap F$ is compact. $\blacksquare$

Exercise 3.3.4. Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \backslash F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^c}$

(a) $K \cap F$ is compact (by last lemma).
$\Rightarrow$ closed also (compact = closed & bounded).

(b) $\overline{F^c \cup K^c}$
$\Rightarrow$ definitely closed, because the closure of any set is closed.
Since K is bdd, K^c is unbounded, so $F^c \cup K^c$ is also unbdd, so $\overline{F^c \cup K^c}$ is also unbdd.

Thus, it is never compact.

(c) $K \setminus F = \{x \in K : x \notin F\}$.

$= K \cap F^c$ ← always bdd, since K is bounded.

$K \cap F^c$ is sometimes closed: (+ bdd $\Rightarrow$ compact)

(eg $K = [0,1]$, $F = [2,\infty)$, $F^c = (-\infty, 2)$

$K \cap F^c = K$, closed.)

and sometime not closed (thus not compact)

[eg. $K = [0,1]$, $F = [0, \frac{1}{2}]$

$F^c = (-\infty, 0) \cup (\frac{1}{2}, \infty)$ $K \cap F^c = (\frac{1}{2}, 1]$ not closed)

($\frac{1}{2}$ is a limit point but not in $K \cap F^c$).

(d) $\overline{K \cap F^c}$ ← (always) closed. ✓

K is bdd (say $|x| \le M$ $\forall x \in K$). $\Leftrightarrow -M \le x \le M$.

Thus $K \cap F^c$ is also bdd. $\subseteq K$.

Every limit pt of $K \cap F^c$ is

the limit of a seq (x_n) of pts. in $K \cap F^c$,
so since $-M \leq x_n \leq M$,

$-M \leq L \leq M$ by OLT.

Thus $y \in \overline{K \cap F^c} \Rightarrow -M \leq y \leq M$.

$\Rightarrow \overline{K \cap F^c}$ is bdd.

Short proof. $K \cap F^c \subseteq [-M, M]$. closed

$\overline{K \cap F^c}$ is the smallest closed set containing $K \cap F^c$, so

$\subseteq [M, M]$. ☐

Exercise 3.3.5. Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

T/F

(a) The arbitrary intersection of compact sets is compact. 5/0

(b) The arbitrary union of compact sets is compact. 2/3

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact. 1.5/3.5

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. 3/2

$F_j = [j, \infty)$

$F_1 = [1, \infty)$

$F_2 = [2, \infty)$

$\vdots$

$\bigcap F_j = \emptyset$.